Special Issue (2003), S153–S157

Advances in Geometry

16-dimensional compact projective planes with 3 fixed points

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Dedicated to Professor Adriano Barlotti on the occasion of his 80th birthday

Let $\mathcal{P} = (P, \mathfrak{L})$ be a topological projective plane with a compact point set P of finite (covering) dimension $d = \dim P > 0$. A systematic treatment of such planes can be found in the book *Compact Projective Planes* [15]. Each line $L \in \mathfrak{L}$ is homotopy equivalent to a sphere $\mathbb{S}_\ell$ with $\ell \mid 8$, and $d = 2\ell$, see [15] (54.11). In all known examples, L is in fact homeomorphic to $\mathbb{S}_\ell$. Taken with the compact-open topology, the automorphism group $\Sigma = \operatorname{Aut}\mathcal{P}$ (of all continuous collineations) is a locally compact transformation group of P with a countable basis, the dimension $\dim\Sigma$ is finite [15] (44.3 and 83.2).

The classical examples are the planes $\mathcal{P}_{\mathbb{K}}$ over the three locally compact, connected fields $\mathbb{K}$ with $\ell = \dim\mathbb{K}$ and the 16-dimensional Moufang plane $\mathcal{O} = \mathcal{P}_{\mathbb{O}}$ over the octonion algebra $\mathbb{O}$. If $\mathcal{P}$ is a classical plane, then $\operatorname{Aut}\mathcal{P}$ is an almost simple Lie group of dimension C_ℓ, where $C_1 = 8$, $C_2 = 16$, $C_4 = 35$, and $C_8 = 78$.

In all other cases, $\dim\Sigma \leqslant \frac{1}{2}C_\ell + 1 \leqslant 5\ell$. Planes with a group of dimension sufficiently close to $\frac{1}{2}C_\ell$ can be described explicitly. More precisely,

the classification program seeks to determine all pairs $(\mathcal{P}, \Delta)$, where Δ is a connected closed subgroup of $\operatorname{Aut}\mathcal{P}$ *and $b_\ell \leqslant \dim\Delta \leqslant 5\ell$ for a suitable bound $b_\ell \geqslant 4\ell - 1$.*

This has been accomplished for $\ell \leqslant 2$ and also for $b_4 = 17$. Here, the case $\ell = 8$ will be studied; the value of b_ℓ varies with the configuration of the fixed elements of Δ.

Most theorems that have been obtained so far require additional assumptions on the structure of Δ. If $\dim\Delta \geqslant 27$, then Δ is always a Lie group [12].

By the structure theory of Lie groups, there are 3 possibilities: (i) Δ is semi-simple, or (ii) Δ contains a central torus subgroup, or (iii) Δ has a minimal normal vector subgroup, cf. [15] (94.26). The first two cases are understood fairly well:

(a) *If Δ is semi-simple and* $\dim\Delta > 28$, *then* $\Delta \cong \mathrm{SL}_3\mathbb{H}$ *and $\mathcal{P}$ is a Hughes plane* (as described in [15] §86), *or* $\Delta \cong \mathrm{Spin}_9(\mathbb{R}, r)$ *with $r \leqslant 1$, or* $\mathcal{P} \cong \mathcal{O}$, see [10], [11].

(b) *If Δ contains a central torus, and if* $\dim\Delta > 30$, *then* $\Delta' \cong \mathrm{SL}_3\mathbb{H}$, see [13].

A group Δ of type (iii) fixes a point or a line, cf. [3] (XI.10.19). Hence (a) and (b) imply

(c) *If* $\dim \Delta > 30$ *and* Δ *has no fixed element, then* $\mathscr{P}$ *is a Hughes plane or* $\mathscr{P} \cong \mathscr{O}$.

The case that Δ fixes exactly one element has been treated in [14]:

(d) *If* $\dim \Delta \geqslant 35$ *and if* Δ *fixes one line and no point, then* $\mathscr{P}$ *is a translation plane.*

All such planes have been determined in [6], [7], [9]. Either $\mathscr{P} \cong \mathscr{O}$ or $\dim \Delta = 35$.

Little progress has been made in the cases where Δ fixes exactly two elements, necessarily a point and a line. If $\dim \Delta \geqslant 40$, then $\mathscr{P}$ and its dual are translation planes [15] (87.7). All *translation planes* with $\dim \Delta \geqslant 38$ are described in [15] (82.28).

(e) *If* $\dim \Delta \geqslant 34$ *and* Δ *fixes exactly* 2 *points and only one line, then* Δ *contains a translation group of dimension at least* 15.

(f) *If* $\dim \Delta \geqslant 33$ *and* Δ *fixes* 2 *points and* 2 *lines, then* Δ *contains a translation group* $\mathrm{T} \cong \mathbb{R}^8$ *and a compact subgroup* $\Phi \cong \mathrm{Spin}_8\mathbb{R}$.

A method to construct all planes with exactly 2 fixed points have been given in [8].

A smaller dimension of Δ suffices if Δ fixes more than two points (the last case to be considered):

Theorem. *If* $\dim \Delta \geqslant 32$ *and* Δ *has (at least)* 3 *fixed points, then* Δ *contains a transitive translation group* T. *Either* $\dim \Delta = 32$ *and a maximal semi-simple subgroup* Ψ *of* Δ *is isomorphic to* $\mathrm{SU}_4\mathbb{C}$, *or* $\dim \Delta \geqslant 37$ *and* $\mathscr{P} \cong \mathscr{O}$.

Translation planes with a group $\Psi \cong \mathrm{SU}_4\mathbb{C}$ have already been studied in [5]. Examples of proper translation planes such that $\mathrm{T}\Psi$ has a fixed point set $S \approx \mathbb{S}_2$ are given in [6].

According to the *stiffness* result [15] (83.23), the stabilizer Λ of a non-degenerate quadrangle satisfies $\dim \Lambda \leqslant 14$. The proof of the theorem depends decisively on Bödi's improvement [1] of the stiffness theorem:

($\square$) *If the fixed elements of the connected Lie group* Λ *form a connected subplane* $\mathscr{E}$, *then* Λ *is isomorphic to the* 14*-dimensional compact group* G_2 *or its subgroup* $\mathrm{SU}_3\mathbb{C}$ *or* $\dim \Lambda < 8$. *If* $\mathscr{E}$ *is a Baer subplane* ($\dim \mathscr{E} = 8$), *then* Λ *is a subgroup of* $\mathrm{SU}_2\mathbb{C}$.

Corollary. *From* $\dim \Lambda > 8$ *it follows that* $\dim \mathscr{E} = 2$.

Proof. Assume that $\dim \mathscr{E} = 4$. If L is any line of $\mathscr{E}$ and if $c \in L \backslash \mathscr{E}$, then $\dim \Lambda_c > 0$ and the fixed elements of Λ_c form a Baer subplane $\langle \mathscr{E}, c \rangle$. Hence $\dim \Lambda_c \leqslant 3$ and $\dim \Lambda \leqslant 11$. An alternative proof is given by [15] (96.35). $\square$

Proof of the Theorem. 1) For any closed subgroup $\Gamma \leqslant \Delta$ and any point x the dimension formula $\dim \Gamma = \dim \Gamma_x + \dim x^\Gamma$ holds, see [15] (96.10). This fact will be used repeatedly without mention.

2) By the stiffness theorem, the stabilizer ∇ of a triangle satisfies $\dim \nabla \leqslant 30$. Hence

all fixed points of Δ are incident with the same line W. There are at least 3 fixed points $u, v, w \in W$ and the stiffness theorem implies $\dim \Delta \leqslant 38$.

3) Because of results (a) and (b), the group Δ has a minimal normal subgroup $\Theta \cong \mathbb{R}^t$. Choose $a \notin W$ and $\varrho \in \Pi \leqslant \Theta$ such that $\Pi \cong \mathbb{R}$ and $a^\varrho \neq a$. Since Δ acts linearly on Θ, the centralizer $\mathrm{Cs}\,\varrho$ is also the centralizer of Π, and the dimension formula gives $\dim \mathrm{Cs}\,\Pi \geqslant 32 - t$. The connected component Λ of $\Delta_a \cap \mathrm{Cs}\,\Pi$ fixes the orbit a^Π pointwise, and the fixed elements of Λ form a connected subplane $\mathscr{E}$, see [15] (42.1). By ($\square$) we have $\dim \Delta_a - t \leqslant \dim \Lambda \leqslant 14$ and $t \geqslant 2$; moreover, $\dim \Lambda = 14$ or $\dim \Lambda \leqslant 8$.

4) Assume first that $t < 8$. Then $\Lambda \cong G_2$ is compact. Remember that the action of any compact or semi-simple Lie group on a real vector space is completely reducible ([2] (35.4)). Each irreducible module of G_2 on $\mathbb{R}^{16}$ has a dimension divisible by 7, see [15] (95.10). Since $\Pi^\Lambda = \Pi$, it follows from $t \leqslant 7$ that the commutator $[\Lambda, \Theta]$ is trivial.

5) The last statement implies that the orbit a^Θ is contained in $\mathscr{E}$. Because Θ is commutative, Θ_a fixes each point of a^Θ. Hence Θ_a acts trivially on the subplane $\mathscr{E}$ generated by a^Π and u, v, w, and the connected component of Θ_a is contained in Λ, but Λ is simple and $\Lambda \cap \Theta = \mathbb{1}$. Therefore, $\dim \Theta_a = 0$ and $\dim a^\Theta = t = 2$.

6) Denote the connected component of Δ_a by ∇. From steps 3) and 5) it follows that $\dim \nabla = 16$. Consequently, ∇ has a 2-dimensional radical $\mathrm{P} = \sqrt{\nabla}$, and $[\Lambda, \mathrm{P}] = \mathbb{1}$. Hence $\mathscr{E}^{\mathrm{P}} = \mathscr{E}$. If c is a point of $\mathscr{E}$ and $c \in aw \setminus \{a, w\}$, then $\dim \mathrm{P}_c > 0$. On the other hand, P_c acts trivially on the smallest closed subplane containing a, c, u, v, and this subplane coincides with $\mathscr{E}$ by [15] (32.7); thus the connected component of P_c would belong to the simple group Λ. This contradiction shows that $t \geqslant 8$.

7) If $t = 8$, then $16 \leqslant \dim \nabla = \dim \varrho^\nabla + \dim \Lambda \leqslant t + 14 = 22$ and $\dim \Lambda \geqslant 8$. Consider the smallest closed subplane $\mathscr{F}$ containing a^Θ and u, v, w, and assume that $\mathscr{P} \neq \mathscr{F} = \mathscr{F}^\nabla$. Then ∇ induces on $\mathscr{F}$ a group ∇/K of dimension $\leqslant 7$, see [15] (83.17). Hence $\dim \mathrm{K} \geqslant 9$ and K contains G_2. The Corollary implies that $\dim \mathscr{F} = 2$ and then $\dim \nabla/\mathrm{K} \leqslant 1$ and $\dim \mathrm{K} > 14$. This contradiction shows $\mathscr{F} = \mathscr{P}$ and $\Theta_a = \mathbb{1}$ (because Θ_a fixes $\mathscr{F}$ pointwise). By ($\square$) there are two possibilities: either $\Lambda \cong G_2$ for some $\varrho \in \Theta$, or $\Lambda \cong SU_3\mathbb{C}$ for each choice of ϱ, and ∇ acts transitively on $\Theta \setminus \{\mathbb{1}\}$ by [15] (96.11). These cases will be treated separately.

8) Suppose that $\Lambda \cong G_2$ and that Λ is contained in the maximal semi-simple subgroup Ψ of Δ. By minimality of Θ and [15] (95.6b), the group Ψ acts irreducibly on Θ and $\Lambda < \Psi$. Clifford's Lemma [15] (95.5) implies that Λ cannot be contained in a proper factor of Ψ, hence Ψ is almost simple. Inspection of the list [15] (95.10) of representations shows that Ψ is locally isomorphic to an orthogonal group. Because each action of $SO_5\mathbb{R}$ on a compact projective plane is trivial ([15] (55.40)), the group Ψ is simply connected and then Ψ has a subgroup $\mathrm{Y} \cong \mathrm{Spin}_7\mathbb{R}$. The central involution $\alpha \in \mathrm{Y}$ cannot be planar (or else Y would induce a group $SO_7\mathbb{R}$ on the fixed plane $\mathscr{F}_\alpha$). Hence α is a reflection with axis W and some center c. Because $\dim \Delta_c \leqslant 22$, we have $\dim c^\Delta \geqslant 10$ and, therefore, $\dim \alpha^\Delta\alpha \geqslant 10$. It is well-known that $\alpha^\Delta\alpha$ is contained in the group T of translations with axis W and that α inverts each translation in T. Consequently, Y acts faithfully on each invariant subgroup of T. There is only one irreducible representation of Y in dimension $\leqslant 16$, viz. the natural one on $\mathbb{R}^8$. It

follows that $\mathrm{T} \cong \mathbb{R}^{16}$ is transitive and that $\dim \mathrm{TY} = 37$. Finally, [4] Satz 3.6 or [15] (81.17) shows that $\mathscr{P} \cong \mathcal{O}$.

9) Consider now the second case mentioned at the end of 7). By [15] (96.19), transitivity of ∇ on $\Theta \backslash \{\mathbb{1}\}$ implies that a maximal compact subgroup Φ of ∇ is transitive on the 7-sphere S consisting of all rays in Θ. We know that $\mathrm{SU}_3\mathbb{C} \cong \Lambda < \Phi$ and that $\dim \Phi < \dim \nabla \leqslant \dim \Lambda + t = 16$. From [15] (96.20–22) we can conclude that the commutator group Φ' is isomorphic to $\mathrm{SU}_4\mathbb{C}$. Let ω denote the central involution in Φ' and note that $\Phi'/\langle\omega\rangle \cong \mathrm{SO}_6\mathbb{R}$. As in step 8), it follows that ω is a reflection with axis W, that the translation group T has dimension at least 10, and that T is the sum of two 8-dimensional irreducible submodules; moreover, $\dim \Delta = \dim \nabla + \dim \mathrm{T} = 32$, and the theorem is proved in the case $t \leqslant 8$.

10) For $t > 8$, the vector group Θ contains a minimal normal subgroup $\mathrm{H} \cong \mathbb{R}^s$ of the connected component Γ of Δ_{av}. Mutatis mutandis, the arguments in steps 3)–9) can be applied to Γ and H instead of Δ and Θ. Using the same notation as before, we have

$$24 \leqslant \dim \Gamma \leqslant \dim a^{\Gamma} + \dim \varrho^{\nabla} + \dim \Lambda \leqslant 8 + s + \dim \Lambda.$$

Hence ($\square$) gives $s \geqslant 2$, moreover, $\Lambda \cong \mathrm{G}_2$ or $s \geqslant 8$.

11) Suppose that $s < 8$. As in step 4), it follows that $[\Lambda, \mathrm{H}] = \mathbb{1}$. Choose a point c in the 2-dimensional subplane $\mathscr{E}$ with $c \in av \backslash \{a, v\}$. Then $\dim c^{\mathrm{H}} \leqslant 1$ and $\mathrm{H}_c \cap \Lambda$ has positive dimension, but Λ is simple. Therefore, $s \geqslant 8$. If $s = 8$, the Theorem is true by the arguments 7)–9).

12) To finish the proof, let $s > 8$ and consider the smallest closed subplane $\mathscr{H}$ containing a^{H} and u, v, w. If k is the dimension of a line of $\mathscr{H}$, then $k \mid 8$. Note that $a^{\mathrm{H}} \subseteq av$ and that H_a induces the identity on $\mathscr{H}$. It follows that $\dim \mathrm{H}_a > 0$, hence $\mathscr{H} \neq \mathscr{P}$ and $k \leqslant 4$. Since H has no compact subgroups other than $\mathbb{1}$, the stiffness theorem ($\square$) shows that $\dim \mathrm{H}_a < 8$, moreover, $\dim \mathrm{H}_a > 3$ implies $k \leqslant 2$. Only the possibility $k = 2$ remains. By [15] (55.4), each closed subplane of $\mathscr{H}$ is connected, and $\mathscr{H}^{\nabla} = \mathscr{H}$ because H is normal in Γ. There are points $b, c \in av \cap \mathscr{H}$ such that $\nabla_{b,c}$ fixes $\mathscr{H}$ pointwise. On the other hand, $\dim \nabla_{b,c} \geqslant 12$. This contradicts the Corollary. $\square$

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Received 28 August, 2002

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